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AN APPROXIMATE METHOD OF CALCULATING THE OSCILLATIONS OF A LIQUID IN INCLINED ELASTIC CAVITIES AND CHANNELS†

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Small oscillations of an ideal incompressible liquid, which partially fills an inclined elastic container (a mobile cavity or channel) of arbitrary form and having a longitudinal plane of symmetry, are considered. The integral condition of continuity of the liquid is obtained by integrating the differential condition of incompressibility and exact satisfaction of the kinematic boundary conditions on the wetted side walls. Using this equation, systems of coordinate functions are constructed which represent the kinematically possible displacements of the liquid, for calculating the oscillations by the Ritz method and the finite-element method. The basic unknown functions, which describe the displacements of the liquid in cross-sections, are approximated by power functions and Legendre functions. Transverse layers of liquid, within the limits of the thickness of which a linear approximation for the unknown functions can be used, are considered as finite elements. © 2004 Elsevier Ltd. All rights reserved.

The Ritz method [1, 2] and the finite-element method [3] are usually employed to solve the problem of the small oscillations of a liquid, partially filling mobile cavities or elastic containers of arbitrary shape. In the case of an incompressible liquid, when solving the problem in displacements it is necessary to satisfy exactly the condition of incompressibility and the kinematic condition that the liquid is always in contact with the surface of the wall during motion, which introduces certain difficulties. Hence, the hydrodynamic pressure (or the potentials of the displacements, the velocities and accelerations of the liquid representing it) is most often considered as the basic unknown and the corresponding variational principle is used, on the basis of which the condition of incompressibility and the kinematic condition on the walls of the cavity when using the Ritz method and the finite-element method are satisfied approximately.

A variational method of solution in displacements of the problem of the oscillations of an incompressible liquid inside an arbitrary shell of revolution was proposed in [4, 5]. By integrating the incompressibility condition and satisfying the kinematic condition on the wetted surface of the shell the problem can be reduced to finding a single unknown function – a longitudinal displacement of the liquid as a function of the axial and radial coordinates. Then, using the Vlasov–Kantorovich method, a system of ordinary differential equations is obtained, and using the Ritz method and the finite-element method, a system of algebraic equations is obtained.

A similar approach was used in [6] to solve, in terms of displacements, the problem of small mainly longitudinal symmetrical oscillations of an incompressible liquid in an inclined elastic container (a cavity or channel) of arbitrary shape with a longitudinal plane of symmetry, where power functions are used to approximate the displacements of the liquid in cross-sections of the cavity. Below we obtain a more accurate solution of this problem in another form using orthogonal Legendre functions to approximate the displacements of the liquid in cross-sections. Using the Ritz method and the finite-element method (transverse layers of the liquid are considered as the finite elements), the problem is reduced to a system of linear algebraic equations.

1. THE INTEGRAL CONDITION OF INCOMPRESSIBILITY OF THE LIQUID

Consider a container (a cavity or channel) with an arbitrary single-closed contour of variable cross-section, partially filled with an ideal incompressible liquid (Fig. 1). We will assume that the container is symmetrical about the xy plane, while the free surface of the liquid $y = H(x)$ is perpendicular to this

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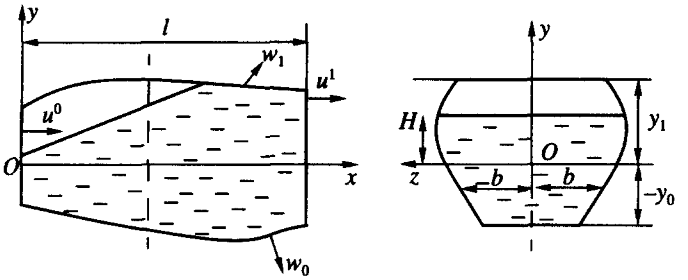

Fig. 1

plane. In the general case, the container has a bottom $y = y_0(x)$, elastic side walls $z = \pm b(x, y)$ and a lid $y = y_1(x)$, the normal displacements of which are denoted by $w_0(x, t)$, $w(x, y, t)$ and $w_1(x, t)$, while the outward unit normals to these surfaces are denoted by $\mathbf{v}_0$, $\mathbf{v}$ and $\mathbf{v}_1$ respectively. If the container is closed at $y = y_0$ and $y = y_1$, i.e. there is no bottom or lid, their transverse dimensions $b_0(x) = b(x, y_0)$ and $b_1(x) = b(x, y_1)$ must be assumed to tend to zero. In the case of oscillations that are symmetrical about the $z = 0$ plane, we will consider only half the cavity when $z \geq 0$. The displacements of the liquid in the directions of the x, y, z axes will be denoted by υ_x, υ_y, υ_z.

The differential condition of incompressibility of the liquid and the kinematic conditions for motion without separation in the plane of symmetry, on the side wall and on the bottom can be written in the form

$$\frac{\partial \upsilon_x}{\partial x} + \frac{\partial \upsilon_y}{\partial y} + \frac{\partial \upsilon_y}{\partial z} = 0 \tag{1.1}$$

$$\upsilon_z = 0 \text{ when } z = 0 \tag{1.2}$$

$$\upsilon_z = \frac{w}{\mathrm{v}_z} + \frac{\partial b}{\partial x}\upsilon_x + \frac{\partial b}{\partial y}\upsilon_y \text{ when } z = b(x,y) \tag{1.3}$$

$$\upsilon_y = \frac{w_0}{\mathrm{v}_{0y}} + y_0'\upsilon_x \text{ when } y = y_0(x) \tag{1.4}$$

where

$$\frac{1}{\mathrm{v}_z} = \sqrt{1 + \left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2}, \quad \frac{1}{\mathrm{v}_{0y}} = -\sqrt{1 + y_0'^2}$$

and v_z, v_{0y} are the projections of the normals $\boldsymbol{\nu}$ and $\boldsymbol{\nu}_0$ on to the z and y axes, respectively.

In the plane of the end walls $x = 0$ and $x = 1$, the normal displacements of which are assumed to be specified, the kinematic boundary conditions have the form

$$\upsilon_x = u^0(y, z, t) \text{ when } x = 0, \quad \upsilon_x = u^1(y, z, t) \text{ when } x = l \tag{1.5}$$

To solve the problem in displacements using the Ritz method or the finite-element method, the condition of incompressibility (1.1) and the kinematic boundary conditions (1.2)–(1.4) must be satisfied exactly. To do this we will reduce them to a single integral equation. Integrating Eq. (1.1) with respect to z and satisfying boundary condition (1.2), we obtain

$$\upsilon_z = -\frac{\partial}{\partial x}\int_0^z \upsilon_x dz - \frac{\partial}{\partial y}\int_0^z \upsilon_y dz \tag{1.6}$$

Substituting this expression into condition (1.3) and then integrating it with respect to y and satisfying condition (1.4), we obtain, after reduction, the integral condition of incompressibility of the liquid, taking the kinematic boundary conditions into account

$$\int_0^b \upsilon_y dz + \frac{\partial}{\partial x}\int_{y_0}^{y}\left(\int_0^b \upsilon_x dz\right)dy + F(x, y, t) = 0 \tag{1.7}$$

where

$$F(x, y, t) = \int_{y_0}^{y} \frac{w}{\nu_z} dy - \frac{w_0}{\nu_{0y}} b_0 \tag{1.8}$$

Consider further the cross-section x = const, in which the container is completely filled with liquid under the lid $y = y_1(x)$. It is necessary to satisfy the following kinematic boundary condition on the lid

$$\upsilon_y = \frac{w_1}{\nu_{1y}} + y_1'\upsilon_x \text{ when } y = y_1(x) \tag{1.9}$$

where

$$\frac{1}{\nu_{1y}} = \sqrt{1 + y_1'^2}$$

and ν_{1y} is the projection of the normal $\boldsymbol{\nu}_1$ onto the y axis.

Taking condition (1.9) into account, when $y = y_1$ we reduce Eq. (1.7) to the form

$$\frac{\partial}{\partial x}\int_{y_0}^{y_1}\left(\int_0^b \upsilon_x dz\right)dy + F(x, y_1, t) + \frac{w_1}{\nu_{1y}} b_1 = 0 \tag{1.10}$$

If this equation is integrated with respect to x, we obtain an equation for the flow rate of the liquid through the cross-section x = const

$$\int_{y_0}^{y_1}\left(\int_0^b \upsilon_x dz\right)dy = Q(x, t) \tag{1.11}$$

where $Q(x, t)$ is the volume of liquid displaced due to normal displacements of the walls in the cutoff part of the cavity.

2. APPROXIMATION OF THE DISPLACEMENTS OF THE LIQUID AND THE CONSTRUCTION OF THE EQUATIONS OF THE OSCILLATIONS

The components of the displacements of the liquid υ_x and υ_y in the case of symmetrical oscillations will be sought in the form (everywhere henceforth summation is carried out over n = 1, 2, ...)

$$\begin{aligned} \upsilon_x &= U_0(x, y, t) + \sum U_n(x, y, t)P_{2n}(\zeta) \\ \upsilon_y &= V_0(x, y, t) + \sum V_n^*(x, y, t)P_{2n}(\zeta) \end{aligned} \tag{2.1}$$

where $\zeta = z/b$; $P_{2n}(\zeta)$ are even Legendre functions.

Equation (1.7), where the displacements υ_x and υ_y are represented in the form (2.1), is satisfied when

$$V_0 = -\frac{1}{b}\left[\frac{\partial}{\partial x}\int_{y_0}^{y} U_0 b dy + F\right], \quad V_n^* = -\frac{1}{b}\frac{\partial}{\partial x}\int_{y_0}^{y} U_n b dy + V_n \tag{2.2}$$

Here U_0, U_1, V_1, U_2, V_2, ... are functions of x, y and t to be determined; to satisfy condition (1.4) it is necessary that

$$V_n = 0 \text{ when } y = y_0(x), \quad n = 1,2,\ldots \tag{2.3}$$

From expression (1.6), taking relations (2.1)–(2.3) into account, the displacement υ_z is defined as

$$\upsilon_0 = \left(U_0\frac{\partial b}{\partial x} + V_0\frac{\partial b}{\partial y} + \frac{w}{\mathrm{v}_z}\right)\zeta + \sum\left(U_n\frac{\partial b}{\partial x} + V_n^*\frac{\partial b}{\partial y}\right)\zeta P_{2n}(\zeta) -$$

$$- \sum\left(\frac{\partial(V_n b)}{\partial y}\int_0^{\zeta} P_{2n}(\zeta)d\zeta\right) \tag{2.4}$$

In the part of the container completely filled with liquid, boundary condition (1.9) on the surface of the lid will be exactly satisfied, if, according to Eq. (1.10), the functions $U_0, U_1, V_1, U_2, V_2, \ldots$ are subject to the conditions

$$\frac{\partial}{\partial x}\int_{y_0}^{y} U_0 b dy + F(x, y_1, t) + \frac{w_1}{\mathrm{v}_{1y}}b_1 = 0, \quad \frac{\partial}{\partial x}\int_{y_0}^{y_1} U_n b dy = 0, \quad n = 1, 2, \ldots \tag{2.5}$$

$$V_n = 0 \text{ when } y = y_0(x); \; V_n = 0 \text{ when } y = y_1(x); \; n = 1,2,\ldots$$

Hence, in the partially and completely filled parts of the container, the main unknowns are the functions $U_0, U_1, V_1, U_2, V_2, \ldots$. They can be sought in the form of polynomials in powers of y

$$\begin{aligned} U_n(x, y, t) &= U_{n0}(x, t) + U_{n1}(x, t)y + U_{n2}(x, t)y^2 + \ldots \\ V_n(x, y, t) &= V_{n0}(x, t) + V_{n1}(x, t)y + V_{n2}(x, t)y^2 + \ldots \end{aligned} \tag{2.6}$$

where $U_{ni}(x, t)$, $V_{ni}(x, t)$ are unknown functions ($n, i = 0, 1, 2, \ldots$); here the highest degree of y must not be less than the highest degree of z of terms retained in series (2.1).

When using the expansion of υ_x in series (2.1) and (2.6) in powers of z and y, boundary conditions (1.5) on the ends $x = 0$ and $x = 1$ can be satisfied exactly, if these ends are undeformed (i.e. if they can only be displaced and turned in the xy plane). If there are elastic plates at these ends, boundary conditions (1.5) can be satisfied approximately using the method of least squares by minimizing the functionals

$$I^0 = \left[\int_{y_0}^{H}\int_0^{b}(\upsilon_x - u^0)^2 dz dy\right]_{x=0}, \quad I^1 = \left[\int_{y_0}^{H}\int_0^{b}(\upsilon_x - u^1)^2 dz dy\right]_{x=l} \quad (H \le y_1)$$

Hence, the functions $U_{ni}(x, t)$ in expansions (2.6) must be determined taking into account the kinematic boundary conditions on the ends, and in the case when part of the cavity is completely filled, conditions (2.5) also.

To solve the hydrodynamic problem (for specified displacements of the walls of the cavity) or the coupled problem of hydroelasticity (when the elastic displacements of the container walls, the bottom, the lid and the ends are unknown) one can use the Ritz method or the finite-element method. The latter is more convenient for calculations in the general case. Here, we consider as finite elements the liquid layers between the cross-sections $x = x_k$ ($k = 0, 1, \ldots N$), bounded by the bottom, the container walls and the free surface of the liquid (or the lid).

For a liquid layer with a free surface, bounded by the cross-sections $x = x_{k-1}$ and $x = x_k$, in the quadratic approximation ($n = 1$), the unknown functions $U_0(x, y, t)$, $U_1(x, y, t)$ and $V_1(x, y, t)$ can be represented in the form

$$\begin{aligned} (U_0, U_1, V_1) &= (R_0^{(k-1)}, R_1^{(k-1)}, R_2^{(k-1)})\alpha_{k-1} + (R_0^{(k)}, R_1^{(k)}, R_2^{(k)})\beta_k \\ R_m^{(k)}(y) &= r_{m0}^{(k)} + r_{m1}^{(k)}y + r_{m2}^{(k)}y^2, \quad m = 0, 1, 2 \\ \alpha_{k-1}(x) &= 1 - \frac{x - x_{k-1}}{x_k - x_{k-1}}, \quad \beta_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} \end{aligned} \tag{2.7}$$

Hence, retaining only two terms with U_0, U_1 and V_0, V_1^* in expansions (2.1), the finite element in the form of a liquid layer with a free surface, taking conditions (2.3) into account, i.e. when

$$V_1(x_{k-1}, y_0(x_{k-1}), t) = 0, \quad V_1(x_k, y_0(x_k), t) = 0$$
$$y_0(x) = y_0(x_{k-1})\alpha_{k-1}(x) + y_0(x_k)\beta_k(x), \quad x_{k-1} \le x \le x_k$$

will have 16 degrees of freedom – 8 at each end, which can be represented by 16 linearly independent generalized coordinates $r_{mi}^{(k-1)}(t)$ and $r_{mi}^{k}(t)$.

For a liquid layer in the completely filled part of the cavity, it is necessary to satisfy conditions (2.5). These additional conditions for the functions $U_0(x, y, t)$, $U_1(x, y, t)$ and $V_1(x, y, t)$ on the left end ($x = x_{k-1}$) and on the right end ($x = x_k$) of the layer can be written in the form

$$\begin{gathered}
\left[\int_{y_0}^{y_1} U_0 b dy\right]_{x = x_k} + \int_{x_{k-1}}^{x_k}\left[F(x, y_1, t) + \frac{w_1}{\mathrm{v}_{1y}}b_1\right]dx = \left[\int_{y_0}^{y_1} U_0 b dy\right]_{x = x_{k-1}} \\
\left[\int_{y_0}^{y_1} U_1 b dy\right]_{x = x_k} = \left[\int_{y_0}^{y_1} U_1 b dy\right]_{x = x_{k-1}} \\
V_1(x_{k-1}, y_0(x_{k-1}), t) = 0, \quad V_1(x_k, y_0(x_k), t) = 0 \\
V_1(x_{k-1}, y_1(x_{k-1}), t) = 0, \quad V_1(x_k, y_1(x_k), t) = 0
\end{gathered} \tag{2.8}$$

Using these 6 conditions we can eliminate 6 generalized coordinates, for example, $r_{00}^{(k)}, r_{10}^{(k)}, r_{20}^{(k-1)}, r_{21}^{(k-1)}$, $r_{20}^{(k)}, r_{21}^{(k)}$. As a result, for the quadratic approximation of υ_x and υ_y with respect to the coordinates z and y of the liquid layer in the completely filled part of the container, we will have 12 degrees of freedom – seven on the left end and five on the right end.

If, in the version of the finite-element method considered, we use the three-term approximations (2.1) for υ_x and υ_y, where U_0, U_1, V_1, U_2, and V_2 are complete polynomials in y up to the fourth power inclusive, and the linear approximation in x within the limits of the layer thickness, the liquid layer with a free surface will have 46 degrees of freedom – 23 at each end, while a liquid layer in the completely filled part of the container will have 39 degrees of freedom – 21 at the left end and 18 at the right end. If the geometrical parameters of an extended container (a channel) and its normal displacements vary fairly slowly along the contour and along the length, then, instead of relations (2.1) and (2.2) we can use for the calculation a single-term approximation with a linear approximation of υ_x along y

$$\begin{gathered}
\upsilon_x = U_0(x, y, t) = U_{00}(x, t) + U_{01}(x, y)y \\
\upsilon_y = V_0(x, y, t), \quad \upsilon_z = \left(U_0\frac{\partial b}{\partial x} + V_0\frac{\partial b}{\partial y} + \frac{w}{\mathrm{v}_z}\right)\zeta
\end{gathered} \tag{2.9}$$

This approximation corresponds to the hypothesis of plane cross-sections of the liquid: for preferentially longitudinal oscillations of the liquid in a symmetrical cavity its cross-sections x = const are displaced in a longitudinal direction and are rotated, remaining plane. They are then deformed in its plane in accordance with the displacements υ_y and υ_z, which are determined from formulae (2.9) taking the first condition of (2.2) into account, so that the condition of incompressibility of the liquid and the kinematic boundary conditions on the wetted surface of the container are satisfied exactly.

When using approximation (2.9) the liquid layer with a free surface has 4 degrees of freedom, characterized by the generalized coordinates $r_{00}^{(k-1)}$, $r_{01}^{(k-1)}$, $r_{00}^{(k)}$, $r_{01}^{(k)}$ (see (2.7)). The liquid layer in the completely filled part of the container in this case, taking the first formula of (2.8) into account, has only 3 degrees of freedom, characterized by the generalized coordinates $r_{00}^{(k-1)}$, $r_{01}^{(k-1)}$, $r_{01}^{(k)}$.

When setting up the equations of the oscillations of a liquid in a fixed or mobile cavity or in an elastic container in generalized coordinates using the Ritz method or the finite-element method, expressions are used for the kinetic and potential energy of the liquid, which in this case can be written in the form

$$\begin{gathered}
T_* = 2\frac{\rho}{2}\int_0^l\int_{y_0}^H\int_0^b(\dot{\upsilon}_x^2 + \dot{\upsilon}_y^2 + \dot{\upsilon}_z^2)dzdydx, \quad H(x) \le y_1(x) \\
\Pi_* = 2\frac{\rho g}{2}\int_0^l\int_0^{b_H}[\upsilon_y - H'\upsilon_x]_{y=H}^2\frac{dzdx}{\sqrt{1 + H'^2}}, \quad b_H(x) = b(x, H)
\end{gathered} \tag{2.10}$$

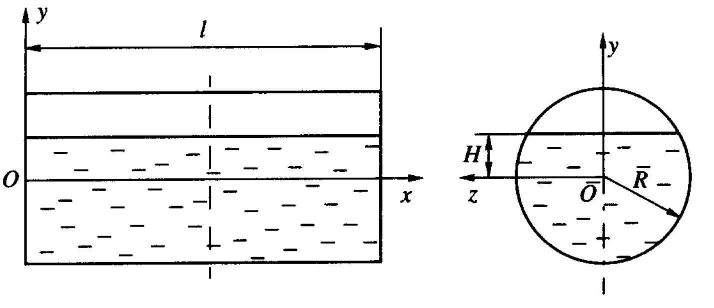

Fig. 2

Here ρ is the liquid density and g is the acceleration due to gravity, the vector of which is perpendicular to the free surface of the liquid. If part of the container is completely filled ($H = y_1$), the potential energy of gravitational waves of the liquid in this part is equal to zero.

3. EXAMPLES OF THE CALCULATION OF THE NATURAL OSCILLATIONS OF THE LIQUID

A cavity in the form of a rectangular horizontal channel. In this case the problem of the longitudinal oscillation of a liquid is plane. The longitudinal displacement for the lowest oscillation mode, according to expressions (2.6) taking the boundary conditions $v_x = 0$ when $x = 0$ and $x = 1$ into account, can be represented in the form

$$v_x = U_0 = (r_0 + r_1 y + r_2 y^2 + \ldots)\sin(\pi x/l)\sin\omega t$$

where $r_0, r_1, r_2, \ldots$ are unknown coefficients.

The values of the square of the dimensionless lowest frequency $\Omega_1^2 = \omega_1^2 H/g$, obtained by the Ritz method in the single-, two- and three-term approximations for a depth $H = 2l/\pi$, are equal to 1.7143, 1.9091 and 1.92776 respectively; the exact solution for the potential motion of the liquid gives a values of 1.92805. Note that the single-term approximation corresponds here to long-wave theory [7], while the two-term approximation corresponds to the hypothesis of plane sections of liquid.

A cavity in the form of a horizontally placed circular cylinder (Fig. 2). The solution by the Ritz method corresponding to the hypothesis of plane sections of liquid (2.9), for the lowest oscillation mode will be sought in the form

$$v_x = U_0 = (r_0(t) + r_1(t)y)\sin(\pi x/l)$$

This solution, for the given theoretical model, is exact. Together with this we will construct a numerical solution of this problem in the same formulation by the finite-element (layers) method. The cavity is divided into N similar layers, perpendicular to the x axis. Within each layer a solution is obtained in the form (2.9) using linear approximations with respect to x for the functions $U_{00}(x, t)$ and $U_{01}(x, t)$. In this case, longitudinal displacements and angles of rotation of the plane sections of the liquid, separating the layers, are considered as the generalized coordinates.

In Table 1 we present the results of calculations of the square of the dimensionless lowest frequency $\omega_1^2 R/g$ for $l = 2R$ and different depths of filling using finite-element method based on the hypothesis of plane sections of the liquid with $N = 4, 8, 16$; we also show the exact solution in displacements using the hypothesis of plane sections of the liquid and the solution obtained by the Ritz method for potential motion of the liquid [2].

A circular cylindrical cavity with an inclined free surface of the liquid. This case corresponds to an inclined cylindrical cavity with a horizontal liquid surface. The left part of the cavity is completely filled while the right part is partially filled (Fig. 3). The depth of filling when the free surface is perpendicular to the axis of the cavity is $a = 2R$.

Table 2 shows values of square of the dimensionless lowest frequency of the oscillations of a liquid in a cavity $\omega_1^2 R/g$ for different angles of inclination of the free surface β, obtained using the finite-element (layers) method based on the hypothesis of plane sections of the liquid with $N = 6$ and $N = 12$; we also show the results of a calculation by the Ritz method for the potential motion of the liquid [2].

Table 1

Method		$H/R = -0.5$	0	0.5
	$N = 4$	0.808	1.335	1.658
FEM	$N = 8$	0.781	1.300	1.625
	$N = 16$	0.774	1.291	1.617
Exact solution		0.762	1.287	1.604
Ritz method		0.763	1.288	1.635

Table 2

Method		$\beta = 30°$	40°	50°	60°
FEM	$N = 6$	1.233	1.092	0.780	0.472
	$N = 12$	1.384	1.102	0.793	0.486
Ritz method		1.408	1.118	0.800	0.492

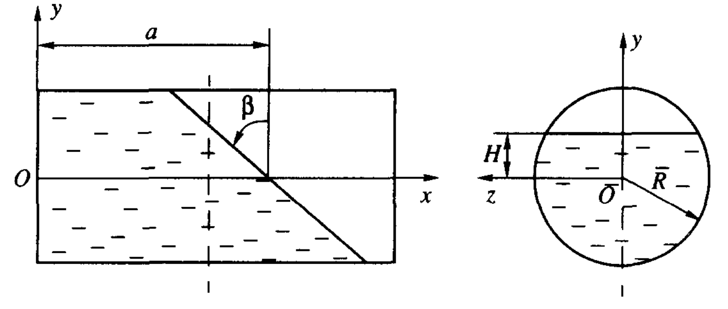

Fig. 3

The results obtained for the natural oscillations of the liquid in symmetrical cavities show that the simplest approximation of the longitudinal displacements of the liquid, corresponding to the plane-section hypothesis, gives completely acceptable accuracy in calculations of the lowest frequency.

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